R. Tauraso (Italy), N. S. Thornber, E. I. Verriest, Z. Vőrős (Hungary), T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), the Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

The Power of Minima

11919 [2016, 613]. *Proposed by Arkady Alt, San Jose, CA.* For positive integers m and k with $k \geq 2$, prove

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} ((n+1)^i - n^i) \sum_{j=1}^{n} j^{k+m-1}.$$

Solution by Pierre Lalonde, Kingsey Falls, QC, Canada. For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. The number of k-tuples $(i_1, \ldots, i_k) \in [n]^k$ such that $\min\{i_1, \ldots, i_k\} \geq r$ is $(n - r + 1)^k$, so the number of k-tuples such that $\min\{i_1, \ldots, i_k\} = r$ is $(n - r + 1)^k - (n - r)^k$. Thus

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{r=1}^{n} ((n - r + 1)^k - (n - r)^k) r^m.$$

Break this expression into two sums, setting $j = n - r + 1$ in the first sum and $j = n - r$ in the second. After recombining the summations, apply the binomial theorem to each term in the first factor and then interchange the order of summation. This gives

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{r=1}^{n} (n - r + 1)^k r^m - \sum_{r=1}^{n} (n - r)^k r^m$$

$$= \sum_{j=1}^{n} j^k (n - j + 1)^m - \sum_{j=1}^{n} j^k (n - j)^m = \sum_{j=1}^{n} ((n + 1 - j)^m - (n - j)^m) j^k$$

$$= \sum_{j=1}^{n} \sum_{i=0}^{m} \binom{m}{i} \left((n+1)^i - n^i\right) (-j)^{m-i} j^k = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) \sum_{j=1}^{n} j^{k+m-i},$$

as desired.

Editorial comment. The identity also holds for $k = 1$. Ramya Dutta noted that the argument above implies

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} (\min\{i_1, \ldots, i_m\})^k,$$

so these sums also equal $\sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} ((n+1)^i - n^i) \sum_{j=1}^{n} j^{k+m-i}$.

Also solved by U. Abel (Germany), T. Amdeberhan & V. H. Moll, D. Beckwith, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), Y. J. Ionin, O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), M. Sawhney, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Generalization of a Fibonacci Identity

11920 [2016, 614]. *Proposed by Ángel Plaza and Sergio Falcón, University of Las Palmas de Gran Canaria, Spain.* For positive integer k, let $\langle F_k \rangle$ be the sequence defined by initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and the recurrence $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$. Find a closed form for $\sum_{i=0}^{n} \binom{2n+1}{i} F_{k,2n+1-2i}$.